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LARGE DEVIATIONS FOR STATISTICS OF JACOBI PROCESS

N. DEMNI ¹ AND M. ZANI ²

ABSTRACT. This paper is aimed to derive large deviations for statistics of Jacobi process already conjectured by M. Zani in her Thesis. To proceed, we write in a more simple way the Jacobi semi-group density. Being given by a bilinear sum involving Jacobi polynomials, it differs from Hermite and Laguerre cases by the quadratic form of its eigenvalues. Our attempt relies on subordinating the process using a suitable random time-change. This will give an analogue of Mehler formula whence we can recover the desired expression by inverting some Laplace transforms. Once we did, an adaptation of Zani's result in the non-steepness case will provide the required large deviations principle.

1. INTRODUCTION

The Jacobi process is a Markov process on $[-1, 1]$ given by the following infinitesimal generator:

$$\mathcal{L} = (1 - x^2) \frac{\partial^2}{\partial^2 x} + (px + q) \frac{\partial}{\partial x}, \quad x \in [-1, 1]$$

for some real p, q , defined up to the first time when it hits the boundary. In fact, it belongs to the class of diffusions associated to some families of orthogonal polynomials, i.e. the infinitesimal generator admits an orthogonal polynomials basis as eigenfunctions ([2]) such as Hermite, Laguerre and Jacobi polynomials. More precisely, if $P_n^{\alpha, \beta}$ denotes the Jacobi polynomial with parameters $\alpha, \beta > -1$ defined by :

$$P_n^{\alpha, \beta}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(-n, n + \alpha + \beta + 1, \alpha + 1; \frac{1 - x}{2} \right), \quad x \in [-1, 1],$$

then we can see that :

$$\mathcal{L} P_n^{\alpha, \beta} = -n(n + \alpha + \beta + 1) P_n^{\alpha, \beta}$$

for $p = -(\beta + \alpha + 2)$ and $q = \beta - \alpha$. In 1964, Wong resolved the forward Kolmogorov or Fokker-Planck equation (see [17], [15])

$$\partial_y^2 [B(y)p] - \partial_y [A(y)p] = \partial_t p, \quad p = p_t(x, y),$$

where B, A are polynomials of degree 2, 1 respectively, and gave the principal solution ($p_0(x, y) = \delta_x(y)$) using the classical Sturm-Liouville theory. This gives rise to a class of

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stationnary Markov processes satisfying :

$$\lim_{t \rightarrow \infty} p_t(x, y) = \int_{x_1}^{x_2} W(x) p_t(x, y) dx = W(y)$$

where W is the density function solution of the corresponding Pearson equation ([17]). In our case, p_t has the discrete spectral decomposition :

$$(1) \quad p_t(x, y) = \left(\sum_{n \geq 0} (R_n)^{-1} e^{-\lambda_n t} P_n^{\alpha, \beta}(x) P_n^{\alpha, \beta}(y) \right) W(y), \quad x, y \in [-1, 1]$$

where

$$\lambda_n = n(n + \alpha + \beta + 1), \quad W(y) = \frac{(1-y)^\alpha (1+y)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}$$

with B denoting the Beta function and ([1], p. 99) :

$$R_n^3 = \|P_n^{\alpha, \beta}\|_{L^2([-1, 1], W(y)dy)}^2 = \frac{\Gamma(\alpha + \beta + 2)}{2n + \alpha + \beta + 1} \frac{(\alpha + 1)_n (\beta + 1)_n}{\Gamma(\alpha + \beta + n + 1)n!}$$

Few years later, Gasper ([10]) showed that this bilinear sum is the transition kernel of a diffusion and that is a solution of the heat equation governed by a Jacobi operator, generalizing a previous result of Bochner for ultraspherical polynomials ([6]). It is worthnoting that λ_n has a quadratic form while in the Hermite (Brownian) and Laguerre (squared Bessel) cases $\lambda_n = n$. Hence, we will try to subordinate the Jacobi process by the mean of a random time-change in order to get a Mehler type formula. What is quite interesting is that subordinated Jacobi process semi-group, say $q_t(x, y)$, is the Laplace transform of $p_{2/t}(x, y)$. Thus, we recover a suitable expression for $p_t(x, y)$ by inverting some Laplace transforms already computed by Biane, Pitman and Yor (see [4], [14]). This will allows us to derive a LDP for the maximum likelihood estimate (MLE) for p in the ultraspherical case, i. e. $q = 0$ ($\beta = \alpha$), a fact conjectured by Zani in her thesis. Then, using a skew product representation of the Jacobi process involving squared Bessel processes, we construct a family $\{\hat{\nu}_t\}_t$ of estimators for the index ν of the squared Bessel process based on a Jacobi trajectory observed till time t . This satisfies a LDP with the same rate function derived for the MLE based on a squared Bessel trajectory.

1.1. Inverse Gaussian subordinator. By an *inverse Gaussian subordinator*, we mean the process of the first hitting time of a Brownian motion with drift $B_t^\mu := B_t + \mu t$, $\mu > 0$, namely,

$$T_t^{\mu, \delta} = \inf\{s > 0; \quad B_s^\mu = \delta t\}, \quad \delta > 0.$$

Using martingale methods, we can show that for each $t > 0$, $u \geq 0$,

$$\mathbb{E}(e^{-u T_t^{\mu, \delta}}) = e^{-t\delta(\sqrt{2u + \mu^2} - \mu)}$$

whence we recover the density below :

$$\nu_t(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-3/2} \exp\left(-\frac{1}{2}\left(\frac{t^2 \delta^2}{s} + \mu^2 s\right)\right) \mathbf{1}_{\{s > 0\}}$$

³ $(P_n^{\alpha, \beta}(x))_{n \geq 0}$ are normalized such that they form an orthogonal basis with respect to the probability measure $W(y)dy$ which is not the same used in [1].

1.2. The subordinated Jacobi Process. Let us consider a Jacobi process $(X_t)_{t \geq 0}$. Then, using (1), the semi-group of the subordinated Jacobi process $(X_{T_t^{\mu, \delta}})_{t \geq 0}$ is given by:

$$\begin{aligned} q_t(x, y) &= \int_0^\infty p_s(x, y) \nu_t(s) ds \\ &= W(y) \sum_{n \geq 0} (R_n)^{-1} \left(\int_0^\infty e^{-\lambda_n s} \nu_t(s) ds \right) P_n^{\alpha, \beta}(x) P_n^{\alpha, \beta}(y) \\ &= W(y) \sum_{n \geq 0} (R_n)^{-1} \mathbb{E}(e^{-\lambda_n T_t^{\mu, \delta}}) P_n^{\alpha, \beta}(x) P_n^{\alpha, \beta}(y) \end{aligned}$$

Writing $\lambda_n = (n + \gamma)^2 - \gamma^2$ where $\gamma = \frac{\alpha + \beta + 1}{2}$, and substituting $\delta = 1/\sqrt{2}$, $\mu = \sqrt{2}\gamma$ for $\alpha + \beta > -1$ in the expression of ν_t , one gets :

$$\mathbb{E}(e^{-\lambda_n T_t^{\mu, \delta}}) = e^{-nt}$$

so that

$$q_t(x, y) = W(y) \sum_{n \geq 0} (R_n)^{-1} e^{-nt} P_n^{\alpha, \beta}(x) P_n^{\alpha, \beta}(y)$$

The last sum has been already computed ([1], p. 385) :

$$\begin{aligned} \sum_{n=0}^\infty (R_n)^{-1} P_n^{\alpha, \beta}(x) P_n^{\alpha, \beta}(y) r^n &= \frac{1-r}{(1+r)^a} \sum_{m, n \geq 0} \frac{\left(\frac{a}{2}\right)_{m+n} \left(\frac{a+1}{2}\right)_{m+n}}{(\alpha+1)_m (\beta+1)_n} \frac{u^m v^n}{m! n!} \\ (2) \quad &= \frac{1-r}{(1+r)^a} F_4\left(\frac{a}{2}, \frac{a+1}{2}, \alpha+1, \beta+1; u, v\right) \end{aligned}$$

where $|r| < 1$, $a = \alpha + \beta + 2$, F_4 is the Appell function and

$$u = \frac{(1-x)(1-y)r}{(1+r)^2} \quad v = \frac{(1+x)(1+y)r}{(1+r)^2}.$$

Then, we use the integral representation of F_4 (see [9], p 51) to get:

$$\begin{aligned} q_t(x, y) &= \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^\infty s^{a-1} e^{-s} {}_0F_1(\alpha+1; \frac{u}{4}s^2) {}_0F_1(\beta+1; \frac{v}{4}s^2) ds \\ &\stackrel{(1)}{=} \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^\infty s^{a-1} e^{-s} \sum_{n \geq 0} \frac{P_n^{\alpha, \beta}(z)}{(\alpha+1)_n (\beta+1)_n} A^n s^{2n} ds \\ &\stackrel{(2)}{=} \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \sum_{n \geq 0} \frac{\Gamma(2n+a)}{(\alpha+1)_n (\beta+1)_n} P_n^{\alpha, \beta}(z) A^n \end{aligned}$$

where in (1), we used (see [12], p 214)

$${}_0F_1(c; w(1-r)/2) {}_0F_1(d; w(1+r)/2) = \sum_{n \geq 0} \frac{P_n^{\alpha, \beta}(r)}{(c)_n (d)_n} w^n, \quad \alpha = c-1, \quad \beta = d-1,$$

in (2), we used Fubini Theorem, $z = \frac{x+y}{1+xy}$ and $A = \frac{(1+xy)r}{2(1+r)^2}$. Letting $r = e^{-t}$, then

$$\begin{aligned} q_t(x, y) &= \frac{W(y)e^{\frac{a-1}{2}t}}{2^{a-1}} \frac{\sinh(t/2)}{(\cosh(t/2))^a} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha, \beta}(z) \left[\frac{(1+xy)}{8 \cosh^2(t/2)} \right]^n \\ &= \frac{W(y) \tanh(t/2) e^{\frac{a-1}{2}t}}{2^{a-1}} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha, \beta}(z) \left[\frac{(1+xy)}{8} \right]^n \left(\frac{1}{\cosh(t/2)} \right)^{2n+a-1}. \end{aligned}$$

Besides,

$$q_t(x, y) = \frac{t e^{\gamma t}}{2\sqrt{\pi}} \int_0^\infty p_s(x, y) s^{-3/2} e^{-\gamma^2 s} e^{-\frac{t^2}{4s}} ds = \frac{t e^{\gamma t}}{2\sqrt{2\pi}} \int_0^\infty p_{2/r}(x, y) r^{-1/2} e^{-2\gamma^2/r} e^{-\frac{t^2}{8}r} dr$$

Thus, noting that $\gamma = (a-1)/2$, we get :

$$\begin{aligned} \int_0^\infty p_{2/r}(x, y) r^{-1/2} e^{-2\gamma^2/r} e^{-\frac{t^2}{8}r} dr &= \frac{\sqrt{2\pi} W(y) \tanh(t/2)}{2^{a-1}} \frac{t/2}{t/2} \\ &\sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha, \beta}(z) \left[\frac{(1+xy)}{8} \right]^n \left(\frac{1}{\cosh(t/2)} \right)^{2n+a-1}. \end{aligned}$$

1.3. The Jacobi semi-group. The following results are due to Biane, Pitman and Yor (see [4], [14]) :

$$(3) \quad \int_0^\infty e^{-\frac{t^2}{8}s} f_{C_h}(s) ds = \left(\frac{1}{\cosh(t/2)} \right)^h, \quad h > 0$$

$$(4) \quad \int_0^\infty e^{-\frac{t^2}{8}s} f_{T_h}(s) ds = \left(\frac{\tanh(t/2)}{(t/2)} \right)^h, \quad h > 0$$

where (C_h) and (T_h) are two families of Lévy processes with respective density functions f_{C_h} and f_{T_h} for fixed $h > 0$. The densities of C_h and T_1 are given by ([4]):

$$\begin{aligned} f_{C_h}(s) &= \frac{2^h}{\Gamma(h)} \sum_{p \geq 0} (-1)^p \frac{\Gamma(p+h)}{p!} f_{\tau(2p+h)}(s) \\ f_{T_1}(s) &= \sum_{k \geq 0} e^{-\frac{\pi^2}{2}(k+\frac{1}{2})^2 s} \mathbf{1}_{\{s>0\}} \end{aligned}$$

where $\tau(c) = \inf\{r > 0; B_r = c\}$ is the Lévy subordinator (i. e, the first hitting time of a standard Brownian motion B) with corresponding density :

$$f_{\tau(2p+h)}(s) = \frac{(2p+h)}{\sqrt{2\pi s^3}} \exp\left(-\frac{(2p+h)^2}{2s}\right) \mathbf{1}_{\{s>0\}}.$$

Thus :

$$p_{2/r}(x, y) = \frac{\sqrt{2\pi r} W(y) e^{2\gamma^2/r}}{2^{a-1}} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha, \beta}(z) \left[\frac{(1+xy)}{8} \right]^n \times (f_{T_1} \star f_{C_{2n+a-1}})(r)$$

or equivalently (where B stands for the Beta function) :

$$p_t(x, y) = \frac{\sqrt{\pi}W(y)}{2^{\alpha+\beta}} \frac{e^{\gamma^2 t}}{\sqrt{t}} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha, \beta}(z) \left[\frac{(1+xy)}{8} \right]^n (f_{T_1} \star f_{C_{2n+\alpha+\beta+1}}) \left(\frac{2}{t} \right).$$

1.4. The ultraspherical case. This case corresponds to $\alpha = \beta > -\frac{1}{2}$ and we will proceed slight differently. Indeed, $a = 2\alpha + 2$ and

$$\begin{aligned} (2) &= \frac{1-r}{(1+r)^{2\alpha+2}} F_4(\alpha+1, \alpha+3/2, \alpha+1, \alpha+1; u, v) \\ &= \frac{1-r}{(1+r)^{2\alpha+2}} \frac{1}{(1-u-v)^{\alpha+3/2}} {}_2F_1\left(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{4uv}{(1-u-v)^2}\right) \end{aligned}$$

where the last equality follows from (see [5])

$$F_4(b, c, b, b; u, v) = (1-u-v)^{-c} {}_2F_1(c/2, (c+1)/2, b; \frac{4uv}{(1-u-v)^2}).$$

Hence,

$$\begin{aligned} q_t(x, y) &= \frac{W(y)e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \frac{\sinh(t)}{(\cosh t - xy)^{\alpha+3/2}} {}_2F_1\left(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{(1-x^2)(1-y^2)}{(\cosh t - xy)^2}\right) \\ &= \frac{W(y)e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \sinh(t) \sum_{n \geq 0} \frac{[(2\alpha+3)/4]_n [(2\alpha+5)/4]_n}{(\alpha+1)_n} \frac{[(1-x^2)(1-y^2)]^n}{(\cosh t - xy)^{2n+\alpha+3/2}}. \end{aligned}$$

Besides, for $h > 0$, we may write :

$$\left(\frac{1}{\cosh t - xy} \right)^h = \sum_{k \geq 0} \frac{(h)_k}{k!} \frac{(xy)^k}{(\cosh t)^{k+h}}$$

since $\left| \frac{xy}{\cosh t} \right| < 1 \quad \forall x, y \in]-1, 1[, \forall t \geq 0$ and where we used:

$$\frac{1}{(1-r)^h} = \sum_{k \geq 0} \frac{(h)_k}{k!} r^k \quad h > 0, |r| < 1.$$

Consequently, using Gauss duplication formula,

$$q_t(x, y) = K_\alpha W(y) e^{\frac{2\alpha+1}{2}t} \tanh(t) \sum_{n, k \geq 0} \frac{\Gamma(\nu(n, k, \alpha) + 1)(xy)^k}{k! n! \Gamma(\alpha + n + 1)} \left[\frac{(1-x^2)(1-y^2)}{4} \right]^n \left(\frac{1}{\cosh t} \right)^{\nu(n, k, \alpha)}$$

where $\nu(n, k, \alpha) = 2n + k + \alpha + 1/2$ and $K_\alpha = \Gamma(\alpha + 1)/[2^{\alpha+1/2}\Gamma(\alpha + 3/2)]$. Thus, since $\gamma = \alpha + 1/2$ when $\alpha = \beta$, one has :

$$\begin{aligned} \int_0^\infty p_s(x, y) s^{-3/2} e^{-\gamma^2 s} e^{-\frac{t^2}{4s}} ds &= \frac{\sqrt{2\pi}\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y) \\ &\quad \sum_{n, k \geq 0} \frac{\Gamma(\nu(n, k, \alpha) + 1)(xy)^k}{k! n! \Gamma(\alpha + n + 1)} \left[\frac{(1-x^2)(1-y^2)}{4} \right]^n \left(\frac{1}{\cosh t} \right)^{\nu(n, k, \alpha)} \end{aligned}$$

or equivalently:

$$\int_0^\infty p_{1/2s}(x, y) e^{-\frac{\gamma^2}{2s}} e^{-\frac{t^2}{2}s} \frac{ds}{\sqrt{s}} = \frac{\sqrt{\pi}\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y) \sum_{n,k \geq 0} \frac{\Gamma(\nu(n, k, \alpha) + 1)(xy)^k}{k!n!\Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4} \right]^n \left(\frac{1}{\cosh t} \right)^{\nu(n, k, \alpha)}$$

Using (3), (4), f_{C_h} et f_{T_1} (we take $t^2/2$ instead of $t^2/8$), the density is written :

$$p_{1/2s}(x, y) = \frac{\sqrt{\pi s}\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)} W(y) e^{\frac{\gamma^2}{2s}} \sum_{n,k \geq 0} \frac{\Gamma(\nu(n, k, \alpha) + 1)}{k!n!\Gamma(\alpha+n+1)} \left(\frac{xy}{2} \right)^k \left[\frac{(1-x^2)(1-y^2)}{4} \right]^n f_{T_1} \star f_{C_{\nu(n, k, \alpha)}}(s)$$

Finally

$$p_t(x, y) = \sqrt{\pi} K_\alpha \frac{e^{\gamma^2 t}}{\sqrt{t}} W(y) \sum_{n,k \geq 0} \frac{\Gamma(\nu(n, k, \alpha) + 1)(xy)^k}{k!n!\Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4} \right]^n f_{T_1} \star f_{C_{\nu(n, k, \alpha)}}\left(\frac{1}{2t}\right)$$

2. APPLICATION TO STATISTICS FOR DIFFUSIONS PROCESSES

2.1. Some properties of the Jacobi process. In the probability scope, we are used to define the Jacobi process on $[-1, 1]$ as the unique strong solution of the SDE :

$$dY_t = \sqrt{1 - Y_t^2} dW_t + (bY_t + c)dt.$$

It is straightforward that $(Y_t)_{t \geq 0} \stackrel{\mathcal{L}}{=} (X_{t/2})_{t \geq 0}$ where X is the Jacobi process already defined in section 1 with $p = 2b$, $q = 2c$. In order to derive some facts, let us make the variable change $y \mapsto (y+1)/2$, this gives up to a time change ($t \rightarrow 4t$) :

$$\begin{aligned} dJ_t &= 2\sqrt{J_t(1-J_t)}dW_t + [2(c-b) + 4bJ_t]dt \\ &= 2\sqrt{J_t(1-J_t)}dW_t + [d - (d+d')J_t]dt \end{aligned}$$

where $d = 2(c-b) = q-p = 2(\beta+1)$ and $d' = -2(c+b) = -(p+q) = 2(\alpha+1)$, which is the Jacobi process of parameters (d, d') already considered in [16]. Moreover, authors provide the following skew-product : let Z_1, Z_2 be two independent Bessel processes of dimensions d, d' and starting from z, z' respectively. Then :

$$\left(\frac{Z_1^2(t)}{Z_1^2(t) + Z_2^2(t)} \right)_{t \geq 0} \stackrel{\mathcal{L}}{=} (J_{A_t})_{t \geq 0}, \quad A_t := \int_0^t \frac{ds}{Z_1^2(s) + Z_2^2(s)}, \quad J_0 = \frac{z}{z+z'}.$$

Using well known properties of squared Bessel processes (see [15]), one deduce that if $d \geq 2$ ($\beta \geq 0$) and $z > 0$, then $J_t > 0$ a. s. for all $t > 0$. Since $1-J$ is still a Jacobi process of parameters (d', d) , then, for $d' \geq 2$, ($\alpha \geq 0$) and $z' > 0$, $J_t < 1$ a. s. for all $t > 0$. These results fit in the one dimensional case those established in [7] for the matrix Jacobi process (Theorem 3. 3. 2, p. 36). Besides, since 0 is a reflecting boundary for Z_1, Z_2 when $0 < d, d' < 2$ ($-1 < \alpha, \beta < 0$), then both 0 and 1 are reflecting boundaries for J .

2.2. LDP in the ultraspherical case. Let us consider the following SDE corresponding to the ultraspherical Jacobi process:

$$(5) \quad \begin{cases} dY_t = \sqrt{1 - Y_t^2} dW_t + bY_t dt \\ Y_0 = y_0 \in]-1, 1[\end{cases}$$

Let $Q_{y_0}^b$ be the law of $(Y_t, t \geq 0)$ on the canonical filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F})$ where Ω is the space of $] -1, 1[$ -valued functions. The parameter b is such that $b \leq -1$ (or $\alpha \geq 0$), so that $-1 < Y_t < -1$ for all $t > 0$. The maximum likelihood estimate of b based on the observation of a single trajectory $(Y_s, 0 \leq s \leq t)$ under Q_0^b is given by

$$(6) \quad \hat{b}_t = \frac{\int_0^t \frac{Y_s}{1 - Y_s^2} dY_s}{\int_0^t \frac{Y_s^2}{1 - Y_s^2} ds}.$$

The main result of this section is the following theorem.

Theorem 1. *When $b \leq -1$, the family $\{\hat{b}_t\}_t$ satisfies a LDP with speed t and good rate function*

$$(7) \quad J_b(x) = \begin{cases} -\frac{1}{4} \frac{(x - b)^2}{x + 1} & \text{if } x \leq x_0 \\ x + 2 + \sqrt{(b - x)^2 + 4(x + 1)} & \text{if } x > x_0 > b \end{cases}$$

where x_0 is the unique solution $x < -1$ of the equation

$$(b - x)^2 = 4x(x + 1).$$

Proof of Theorem 1:

We follow the scheme of Theorem 3.1 in [19]. Let us denote by :

$$S_{t,x} = \int_0^t \frac{Y_s}{1 - Y_s^2} dY_s - x \int_0^t \frac{Y_s^2}{1 - Y_s^2} ds$$

so that for $x > b$ (resp. $x < b$), $P(\hat{b}_t \geq x) = P(S_{t,x} \geq 0)$ (resp. $P(\hat{b}_t \leq x) = P(S_{t,x} \leq 0)$). Therefore, to derive a large deviation principle on $\{\hat{b}_t\}$, we seek a LDP result for $S_{t,x}/t$ at 0. Let us compute the normalized cumulant generating function $\Lambda_{t,x}$ of $S_{t,x}$:

$$(8) \quad \Lambda_{t,x}(\phi) = \frac{1}{t} \log E(e^{\phi S_{t,x}})$$

From Girsanov formula, the generalized densities are given by

$$\frac{dQ_a^b}{dQ_a^{b_0}} = \exp \left\{ (b - b_0) \int_0^t \frac{Y_s}{1 - Y_s^2} dY_s - \frac{1}{2} (b^2 - b_0^2) \int_0^t \frac{Y_s^2}{1 - Y_s^2} ds \right\}$$

From Itô formula,

$$F(Y_t) = -\frac{1}{2} \log(1 - Y_t^2) = \int_0^t \frac{Y_s}{1 - Y_s^2} dY_s + \frac{1}{2} \int_0^t \frac{1 + Y_s^2}{1 - Y_s^2} ds.$$

Let us denote by

$$\mathcal{D}_1 = \{\phi : (b + 1)^2 + 2\phi(x + 1) \geq 0\}.$$

Set $b(\phi, x) = -1 - \sqrt{(b+1)^2 + 2\phi(x+1)}$ for all $\phi \in \mathcal{D}_1$. Then :

$$\Lambda_t(\phi, x) = \frac{1}{t} \log E_{b(\phi, x)}(\exp(\{\phi + b - b(\phi, x)\}[F(Y_t) - F(y_0) - t/2]\})$$

When starting from $y_0 = 0$, the semi-group is deduced from that of X :

$$\tilde{p}_t(0, y) = \sqrt{2\pi} K_\alpha \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{n \geq 0} \frac{\Gamma(2n + \alpha + \frac{3}{2})}{4^n n! \Gamma(n + \alpha + 1)} (1 - y^2)^{n+\alpha} f_{T_1} \star f_{C_{2n+\gamma}}(1/t),$$

where $p = -2(\alpha + 1) = 2b \leq -2$ and $\gamma = -(p + 1)/2 = \alpha + 1/2$. Denote by

$$\mathcal{D} = \{\phi \in \mathcal{D}_1 : G(\phi) = b + b(\phi, x) + \phi < 0\}.$$

For any $\phi \in \mathcal{D}$, the expectation above is finite and a simple computation gives :

$$\begin{aligned} \Lambda_t(\phi, x) &= -\frac{\phi + b - b(\phi, x)}{2} + \frac{1}{t} \log \mathbb{E}_{b(\phi, x)}((1 - Y_t^2)^{-(\phi + b - b(\phi, x))/2}) \\ &= \Lambda(\phi, x) + \frac{1}{t} \log \frac{\sqrt{2\pi} K_{\alpha(\phi, x)} R_t(\phi, x)}{\sqrt{t}} \end{aligned}$$

where

$$R_t(\phi, x) = \sum_{n \geq 0} \frac{\Gamma(2n - b(\phi, x) + 1/2)}{4^n n! \Gamma(n - b(\phi, x))} B\left(n - \frac{\phi + b + b(\phi, x)}{2}, \frac{1}{2}\right) e^{\gamma^2 t/2} f_{T_1} \star f_{C_{2n+\gamma}}\left(\frac{1}{t}\right)$$

and B stands for the Beta function. Moreover, by dominated convergence theorem

$$\lim_{t \rightarrow \infty} \mathbb{E}_{b(\phi, x)}((1 - Y_t^2)^{-(\phi + b - b(\phi, x))/2}) = \int_{-1}^1 (1 - y^2)^{-(\phi + b + b(\phi, x))/2 - 1} dy < \infty$$

for $\phi \in \mathcal{D}$. Hence $\Lambda_t \rightarrow \Lambda$ as $t \rightarrow \infty$. The following lemma details the domain \mathcal{D} of Λ_t :

Lemma 1. *Denote by*

$$\phi_0 = -\frac{(b+1)^2}{2(x+1)}.$$

i) If $x < (b^2 + 3)/2(b - 1)$: then $\mathcal{D} = (-\infty, \phi_0)$.

ii) If $(b^2 + 3)/2(b - 1) < x < -1$: then $\mathcal{D} = (-\infty, \phi_1)$ where ϕ_1 is solution of $G(\phi) = 0$.

iii) If $x > -1$: then $\mathcal{D} = (\phi_0, \phi_1)$.

In case i) of Lemma above, Λ is steep. It achieves its unique minimum in ϕ_m solution of

$$\partial \Lambda / \partial \phi(\phi, x) = 0,$$

i.e. $b(\phi, x) = x$. It is easy to see that

$$\phi_m = \frac{x+1}{2} - \frac{(b+1)^2}{2(x+1)} < \phi_0.$$

Hence, Gärtner-Ellis Theorem gives for $x < b < (b^2 + 3)/2(b - 1)$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log P(\hat{b}_t \leq x) = \lim_{t \rightarrow \infty} \frac{1}{t} \log P(S_{t,x} \leq 0) = \inf_{\phi \in]\infty, \phi_0]} \Lambda(\phi, x) = \Lambda(\phi_m, x) = -\frac{1}{4} \frac{(x-b)^2}{x+1}.$$

If $b < x < (b^2 + 3)/2(b - 1)$, notice that $\phi_m > 0$ and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log P(\hat{b}_t \geq x) = \lim_{t \rightarrow \infty} \frac{1}{t} \log P(S_{t,x} \geq 0) = \inf_{\phi \in (0, \phi_0]} \Lambda(\phi, x) = \Lambda(\phi_m, x) = -\frac{1}{4} \frac{(x - b)^2}{x + 1}.$$

In cases ii) and iii) of Lemma 1, Λ is not steep. Nevertheless, if the infimum of Λ is reached in $\overset{\circ}{\mathcal{D}}$, we can follow the scheme of Gartner–Ellis theorem for the change of probability in the infimum bound. This infimum is reached if and only if

$$(9) \quad \partial \Lambda / \partial \phi(\phi_1, x) > 0.$$

This above condition gives the following cases: denote by x_0 the unique solution $x < -1$ of $g(x) := 4x(x + 1) - (b - x)^2 = 0$. Since g is decreasing on $] -\infty, -1]$ and $g(b^2 + 3/(2(b - 1))) = (3/4)(b + 1)^2 > 0 = g(x_0)$, then $x_0 > (b^2 + 3)/[2(b - 1)]$.

• if $(b^2 + 3)/2(b - 1) < x < x_0 < -1$, the derivative $\partial \Lambda / \partial \phi(\phi_1, x) > 0$, Λ achieves its minimum on ϕ_m and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log P(\hat{b}_t \geq x) = \Lambda(\phi_m, x) = -\frac{(x - b)^2}{4(x + 1)}.$$

• if $x_0 < x < -1$ or $x > -1$, then $\partial \Lambda / \partial \phi(\phi_1, x) < 0$. We apply Theorem 2 of the appendix, which is due to Zani [19]. Let us verify that the assumptions are satisfied. Indeed, the only singularity ϕ_1 of R_t comes from $B(n - [\phi + b + b(\phi, x)]/2, 1/2)$ when $n = 0$, and more precisely, from $\Gamma(-[\phi + b + b(\phi, x)]/2)$. We can write

$$(10) \quad \Lambda_t(\phi, x) = \Lambda(\phi, x) + \frac{1}{t} \log \Gamma\left(-\frac{\phi + b + b(\phi, x)}{2}\right) + \frac{1}{t} \log \frac{\sqrt{2\pi} K_{\alpha(\phi, x)} \tilde{R}_t(\phi, x)}{\sqrt{t}},$$

where

$$(11) \quad \tilde{R}_t(\phi, x) = \frac{R_t(\phi, x)}{\Gamma(-[\phi + b + b(\phi, x)]/2)}$$

Now

$$\forall n \geq 0, \quad \frac{B\left(n - \frac{\phi + b + b(\phi, x)}{2}, \frac{1}{2}\right)}{\Gamma\left(-\frac{\phi + b + b(\phi, x)}{2}\right)}$$

is analytic on some neighbourhood of ϕ_1 . Besides, ϕ_1 is a pole of order one, i.e.

$$\lim_{\phi \rightarrow \phi_1, \phi < \phi_1} \frac{b + \phi + b(\phi, x)}{\phi - \phi_1} = c > 0,$$

and since $\lim_{\rho \rightarrow 0+} \rho \Gamma(\rho) = 1$, we can write

$$\frac{1}{t} \log \Gamma\left(-\frac{\phi + b + b(\phi, x)}{2}\right) = -\frac{\log(\phi_1 - \phi)}{t} + \frac{h(\phi)}{t}.$$

The function h is analytic on \mathcal{D} and can be extended to an analytic function on $] \phi_1 - \xi, \phi_1 + \xi[$ for some positive ξ . Finally, we focus on $\tilde{R}_t(\phi, x)/\sqrt{t}$ and show that it converges

uniformly as $t \rightarrow \infty$. To proceed, we shall prove that this ratio is bounded from above and below away from 0. Setting $A_n(t) := e^{\gamma^2 t/2} f_{T_1} \star f_{C_{2n+\gamma}}(1/t)$, one has :

$$\begin{aligned} \frac{A_n(t)}{\sqrt{t}} &\leq \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{k,l \geq 0} U_{k,n} \int_0^{1/t} \exp -\frac{1}{2} \left[\frac{(2n+2k+\gamma)^2}{s} + \pi^2(l+\frac{1}{2})^2(\frac{1}{t}-s) \right] \frac{ds}{s^{3/2}} \\ &= \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{k,l \geq 0} U_{k,n} \int_t^\infty \exp -\frac{1}{2} \left[(2n+2k+\gamma)^2 s + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts}) \right] \frac{ds}{\sqrt{s}} \\ &< e^{-2n^2} \sum_{k,l \geq 0} U_{k,n} e^{-2k^2} \int_t^\infty \exp -\frac{1}{2} \left[(2n+2k+\gamma)^2(s-t) + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts}) \right] \frac{ds}{\sqrt{ts}} \\ &= e^{-2n^2} \sum_{k,l \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp -\frac{1}{2} \left[(2n+2k+\gamma)^2 s + \pi^2 l^2(\frac{s}{t(t+s)}) \right] \frac{ds}{\sqrt{t(t+s)}} \end{aligned}$$

with

$$U_{k,n} = \frac{\Gamma(2n+k+\gamma) 2^{2n+\gamma} (2n+2k+\gamma)}{k! \Gamma(2n+\gamma)}.$$

Let $\Theta(x) = \sum_{l \in \mathbb{Z}} e^{-\pi l^2 x} = 1 + 2 \sum_{l \geq 1} e^{-\pi l^2 x}$ denote the Jacobi Theta function. Then

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2} \left[\sum_{k \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp - \left[\frac{(2n+2k+\gamma)^2 s}{2} \right] \Theta \left(\frac{\pi s}{2t(t+s)} \right) \frac{ds}{\sqrt{t(t+s)}} + C(n, t) \right]$$

where

$$C(n, t) = \frac{1}{2\sqrt{t}} \sum_{k,l \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp - \left[\frac{(2n+2k+\gamma)^2 s}{2} \right] \frac{ds}{\sqrt{t+s}}$$

Recall that $\Theta(x) = (1/\sqrt{x})\Theta(1/x)$, which yields :

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2} \sum_{k \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp - \left[\frac{(2n+2k+\gamma)^2 s}{2} \right] \Theta \left(\frac{2t(t+s)}{\pi s} \right) \frac{ds}{\sqrt{s}} + \frac{C(n)}{2\sqrt{t}}$$

where

$$C(n) = e^{-2n^2} \sum_{k,l \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp - \left[\frac{(2n+2k+\gamma)^2 s}{2} \right] \frac{ds}{\sqrt{s}}$$

Since $e^{-l^2 z} < e^{-lz}$, then $\Theta(z) \leq 3$ for $z > 1$. Hence, as $2t/\pi \leq 2t(t+s)/(\pi s)$, then for t large enough :

$$\frac{A_n(t)}{\sqrt{t}} < 3e^{-2n^2} \sum_{k \geq 0} U_{k,n} e^{-2k^2} \int_0^\infty \exp - \left[\frac{(2n+2k+\gamma)^2 s}{2} \right] \frac{ds}{\sqrt{s}} + C(n)$$

This gives a lower bound for \tilde{R}_t/\sqrt{t} . Besides,

$$\begin{aligned} \frac{\tilde{R}_t(\phi, x)}{\sqrt{t}} &> \frac{\sqrt{\pi}\Gamma(1/2 - b(\phi, x))}{\Gamma(-b(\phi, x))\Gamma\{[1 - (\phi + b + b(\phi, x))/2]\}} \frac{A_0(t)}{\sqrt{t}} \\ &= C(b, \phi, x) \sum_{k, l \geq 0} (-1)^k V_k \int_0^\infty \exp -\frac{1}{2} \left[(2k + \gamma)^2 s + \pi^2 (l + \frac{1}{2})^2 (\frac{s}{t(t+s)}) \right] \frac{ds}{\sqrt{t(t+s)}} \end{aligned}$$

where $V_k(t) := U_{k,0} e^{-2k(k+\gamma)t}$. One may choose t large enough independent of k such that $V_k(t) \geq V_{k+1}(t)$ for all $k \geq 0$. In fact, such t satisfies :

$$e^{2(2k+\gamma+1)t} \geq e^{2t} \geq \sup_{k \geq 0} \frac{U_{k+1,0}}{U_{k,0}} = \sup_{k \geq 0} \frac{(k+\gamma)(2k+\gamma+2)}{(k+1)(2k+\gamma)}$$

Then :

$$\begin{aligned} \frac{\tilde{R}_t}{\sqrt{t}} &> C(b, \phi, x) [V_0(t) - V_1(t)] \sum_{l \geq 0} \int_0^\infty \exp -\frac{1}{2} \left[\gamma^2 s + \pi^2 (l + \frac{1}{2})^2 (\frac{s}{t(t+s)}) \right] \frac{ds}{\sqrt{t(t+s)}} \\ &> C(b, \phi, x) [\gamma 2^\gamma - V_1(t)] \sum_{l \geq 0} \int_0^\infty \exp -\frac{1}{2} \left[\gamma^2 s + \pi^2 (l + 1)^2 (\frac{s}{t(t+s)}) \right] \frac{ds}{\sqrt{t(t+s)}} \\ &= \frac{C(b, \phi, x)}{2} [\gamma 2^\gamma - V_1(t)] \left\{ \int_0^\infty e^{-\gamma^2 s/2} \Theta \left(\frac{\pi s}{2t(t+s)} \right) \frac{ds}{\sqrt{t(t+s)}} - C(t) \right\}. \end{aligned}$$

where

$$C(t) = \frac{1}{2\sqrt{t}} \int_0^\infty e^{-\gamma^2 s/2} \frac{ds}{\sqrt{t(t+s)}} < c \int_0^\infty e^{-\gamma^2 s/2} \frac{ds}{\sqrt{s}}, \quad c < \sqrt{\frac{2}{\pi}}.$$

for t large enough. Following the same scheme as for the upper bound, one gets :

$$\begin{aligned} \frac{\tilde{R}_t}{\sqrt{t}} &> \frac{C(b, \phi, x)}{2} \gamma 2^\gamma \left\{ \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\gamma^2 s/2} \Theta \left(\frac{2t(t+s)}{\pi s} \right) \frac{ds}{\sqrt{s}} - C(t) \right\} \\ &> \frac{C(b, \phi, x)}{2} \gamma 2^\gamma \left(\sqrt{\frac{2}{\pi}} - c \right) \int_0^\infty e^{-\gamma^2 s/2} \frac{ds}{\sqrt{s}} > 0. \quad \square \end{aligned}$$

As a result,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log P(\hat{b}_t \geq x) = \Lambda(\phi_1, x) = -(x + 2 + \sqrt{(b-x)^2 + 4(x+1)}),$$

which ends the proof of Theorem 1. ■

2.3. Jacobi-squared Bessel processes duality. By Itô's formula and Lévy criterion, one claims that $(Y_t^2)_{t \geq 0}$ is a Jacobi process of parameters $d = 1$, $d' = -2b \geq 2$. Indeed :

$$\begin{aligned} dZ_t := d(Y_t^2) &= 2Y_t dY_t + \langle Y \rangle_t = 2Y_t \sqrt{1 - Y_t^2} dW_t + [(2b-1)Y_t^2 + 1]dt \\ &= 2\sqrt{Z_t(1-Z_t)} \operatorname{sgn}(Y_t) dW_t + [(2b-1)Z_t + 1]dt \\ &= 2\sqrt{Z_t(1-Z_t)} dB_t + [(2b-1)Z_t + 1]dt \end{aligned}$$

Using the skew product previously stated, there exists R , a squared Bessel process of dimension $d' = 2(\nu + 1) = -2b$ and starting from r so that :

$$\hat{\nu}_t := -\hat{b}_t - 1 = \frac{\log(1 - Y_t^2) + t}{2 \int_0^t \frac{Y_s^2}{1 - Y_s^2} ds}$$

is another estimator of ν based on a Jacobi trajectory observed till time t . Set $t = \log u$, then

$$\hat{\nu}_{\log u}^1 = \frac{\log[u(1 - Y_{\log u}^2)]}{2 \int_0^{\log u} \frac{Y_s^2}{1 - Y_s^2} ds} = \frac{\log[u(1 - Y_{\log u}^2)]}{2 \int_1^u \frac{Y_{\log s}^2}{s(1 - Y_{\log s}^2)} ds}$$

and $\{\hat{\nu}_{\log u}^1\}_u$ satisfies a LDP with speed $\log u$ and rate function $J_{-(\nu+1)}(-(x+1))$.

When starting at $R_0 = 1$, the MLE of ν based on a Bessel trajectory is given by (cf [19], p. 132) :

$$\hat{\nu}_t^1 = \frac{\int_0^t \frac{dX_s}{X_s} - 2 \int_0^t \frac{ds}{X_s}}{2 \int_0^t \frac{ds}{X_s}} = \frac{\log(X_t)}{2 \int_0^t \frac{ds}{X_s}}$$

with associated rate function :

$$I_\nu(x) = \begin{cases} \frac{(x-\nu)^2}{4x} & \text{if } x \geq x_1 \\ 1 - x + \sqrt{(\nu - x)^2 - 4x} & \text{if } x < x_1 \end{cases} \quad := \frac{-(\nu+2)+2\sqrt{\nu^2+\nu+1}}{3}$$

A glance at both rate functions gives $I_\nu(x) = J_{-(\nu+1)}(-(x+1))$ and $x_0 = -(x_1 + 1)$.

3. APPENDIX

Let $\{Y_t\}_{t \geq 0}$ be a family of real random variables defined on (Ω, \mathcal{F}, P) , and denote by μ_t the distribution of Y_t . Suppose $-\infty < m_t := EY_t < 0$. We look for large deviations bounds for $P(Y_t \geq y)$. Let Λ_t be the n.c.g.f. of Y_t :

$$\Lambda_t(\phi) = \frac{1}{t} \log E(\exp\{\phi t Y_t\}),$$

and denote by D_t the domain of Λ_t . We assume that there exists $0 < \phi_1 < \infty$ such that for any t

$$\sup\{\phi : \phi \in D_t\} = \phi_1$$

and $[0, \phi_1) \subset D_t$. We assume also that for $\phi \in D$

Assumption 1.

$$(12) \quad \Lambda_t(\phi) = \Lambda(\phi) - \frac{\alpha}{t} \log(\phi_1 - \phi) + \frac{R_t(\phi)}{t}$$

where

- $\alpha > 0$
- Λ is analytic on $(0, \phi_1)$, convex, with finite limits at endpoints, such that $\Lambda'(0) < 0$, $\Lambda'(\phi_1) < \infty$, and $\Lambda''(\phi_1) > 0$.
- R_t is analytic on $(0, \phi_1)$ and admits an analytic extension on a strip $D_\beta^\gamma = (\phi_1 - \beta, \phi_1 + \beta) \times (-\gamma, \gamma)$, where β and γ are independent of t .
- $R_t(\phi)$ converges as $t \rightarrow \infty$ to some $R(\phi)$ uniformly on any compact of D_β^γ .

Theorem 2. *Under 1*

For any $\Lambda'(0) < y < \Lambda'(\phi_1)$,

$$(13) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log P(Y_t \geq y) = - \sup_{\phi \in (0, \phi_1)} \{y\phi - \Lambda(\phi)\}.$$

For any $y \geq \Lambda'(\phi_1)$,

$$(14) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log P(Y_t \geq y) = -y\phi_1 + \Lambda(\phi_1).$$

The rate function is continuously differentiable with a linear part.

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